Exercise 18

Use power series to solve the differential equation

$$y'' - xy' - 2y = 0$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Differentiate the series with respect to x once more.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x and 2 inside the respective summands.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

Because of n in the summand, the second series can start from $n = 0$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

Make the substitution $n = k + 2$ in the first series and the substitution $n = k$ in the second and third series.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^{(k+2)-2} - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} 2 a_k x^k = 0$$

Simplify the first sum.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} 2 a_k x^k = 0$$

Now that all the sums start from $k = 0$ and have x^k in the summand, they can be combined.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} x^k - k a_k x^k - 2 a_k x^k \right] = 0$$

Simplify the summand.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} - (k+2)a_k\right] x^k = 0$$

Since x^k isn't zero, the quantity in square brackets must be zero.

$$(k+2)(k+1)a_{k+2} - (k+2)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{1}{k+1} a_k$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$\begin{aligned}
k=0: &\quad a_2 = \frac{1}{0+1}a_0 = \frac{1}{1}a_0 \\
k=1: &\quad a_3 = \frac{1}{1+1}a_1 = \frac{1}{2}a_1 \\
k=2: &\quad a_4 = \frac{1}{2+1}a_2 = \frac{1}{3}\left(\frac{1}{1}a_0\right) = \frac{1\cdot 1}{3\cdot 1}a_0 \\
k=3: &\quad a_5 = \frac{1}{3+1}a_3 = \frac{1}{4}\left(\frac{1}{2}a_1\right) = \frac{1\cdot 1}{4\cdot 2}a_1 \\
k=4: &\quad a_6 = \frac{1}{4+1}a_4 = \frac{1}{5}\left(\frac{1\cdot 1}{3\cdot 1}a_0\right) = \frac{1\cdot 1\cdot 1}{5\cdot 3\cdot 1}a_0 \\
k=5: &\quad a_7 = \frac{1}{5+1}a_5 = \frac{1}{6}\left(\frac{1\cdot 1}{4\cdot 2}a_1\right) = \frac{1\cdot 1\cdot 1}{6\cdot 4\cdot 2}a_1 \\
&\quad \vdots
\end{aligned}$$

The general formula for the even subscripts is

$$a_{2m} = \frac{1}{(2m-1)!!}a_0 = \frac{2^{m-1}(m-1)!}{(2m-1)!}a_0,$$

and the general formula for the odd subscripts is

$$a_{2m+1} = \frac{1}{(2m)!!}a_1 = \frac{1}{2^m m!}a_1.$$

Therefore, the general solution is

$$\begin{aligned}
y(x) &= \sum_{m=0}^{\infty} a_m x^m \\
&= a_0 + \sum_{m=1}^{\infty} a_{2m}x^{2m} + \sum_{m=0}^{\infty} a_{2m+1}x^{2m+1} \\
&= a_0 + \sum_{m=1}^{\infty} \frac{2^{m-1}(m-1)!}{(2m-1)!}a_0 x^{2m} + \sum_{m=0}^{\infty} \frac{1}{2^m m!}a_1 x^{2m+1},
\end{aligned}$$

where a_0 and a_1 are arbitrary constants.